CHAPTER 2: MATHEMATICAL MODELLING

A) Introduction

- **The first step in analysis and design of a control system is to get the system specification**

- **Next is to get the block diagrams to represent the system operation**

- **Then, the schematic diagrams of subsystems are obtained, where the appropriate assumptions are made at this stage**

- **The following step is to get the mathematical modeling of the control system. For the electrical network modeling – Ohm's Laws, Kirchoff's Laws, etc. For mechanical system – Newton's Laws, etc.**

- **Every subsystem in the overall system can be modeled separately and then combined together to get the overall system model.**

- **For the mathematical modeling, we have to know several basic mathematical concepts.**

- **A system represented by a differential equation is difficult to model as a block diagram. Hence, Laplace Transform is used to get the system's transfer function with all initial conditions are assumed to be zero.**

- **The knowledge of how to use the Laplace Transform table (esp. the theorems) is very useful in modeling a system.**

Item no.	Theorem	Name
1.	$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathscr{L}[kf(t)] = kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathscr{L}[e^{-at}f(t)] = F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}[f(t-T)] = e^{-sT}F(s)$	Time shift theorem
6.	$\mathscr{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathscr{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0-) - \dot{f}(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^nf}{dt^n}\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{t} f(\tau)\,d\tau\right] = \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty) = \lim_{s\to 0} sF(s)$	Final value theorem[1]
12.	$f(0+) = \lim_{s\to\infty} sF(s)$	Initial value theorem[2]

[1] For this theorem to yield correct finite results, all roots of the denominator of $F(s)$ must have negative real parts and no more than one can be at the origin.
[2] For this theorem to be valid, $f(t)$ must be continuous or have a step discontinuity at $t = 0$ (i.e., no impulses or their derivatives at $t = 0$).

- **The concept of the transfer function has been looked into in the previous chapter. It can be obtained by transforming the differential equation into Laplace domain. (assuming zero initial conditions)**

- **For example, find the transfer function *G(s) = C(s)/R(s)* for the differential equation**

$$\frac{d^3c}{dt^3} + 3\frac{d^2c}{dt^2} + 7\frac{dc}{dt} + 5c = \frac{d^2r}{dt^2} + 4\frac{dr}{dt} + 3r$$

$$➔\ G(s) = \frac{C(s)}{R(s)} = \frac{s^2 + 4s + 3}{s^3 + 3s^2 + 7s + 5}$$

B) Modelling of electrical systems

- **Equivalent circuit for electrical networks consists of three passive linear components, which are resistors, capacitors and inductors**

- **The following table gives the components' relationships between voltage and current, and between voltage and charge under zero initial conditions:**

Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)\,d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

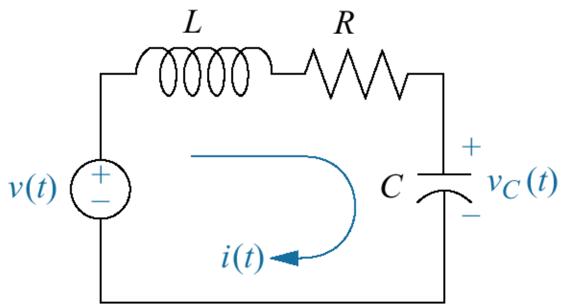
Note: The following set of symbols and units is used throughout this book: $v(t)$ = V (volts), $i(t)$ = A (amps), $q(t)$ = Q (coulombs), C = F (farads), R = Ω (ohms), G = ℧ (mhos), L = H (henries).

- **We can combine those electrical components into circuits, decide the input and output, and then find the transfer function.**

- **We can sum voltage around loops or sum currents at nodes, depending on which technique involves less effort.**

- **From these relations, we can write the differential equations for the circuit, take the Laplace transforms of the differential equations and finally solve for the transfer function.**

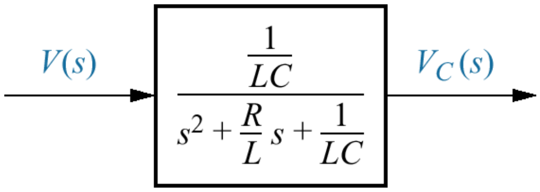
- **Example: Find the transfer function relating the capacitor voltage, $V_c(s)$ to the input voltage, $V(s)$.**



$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(t)dt = v(t)$$

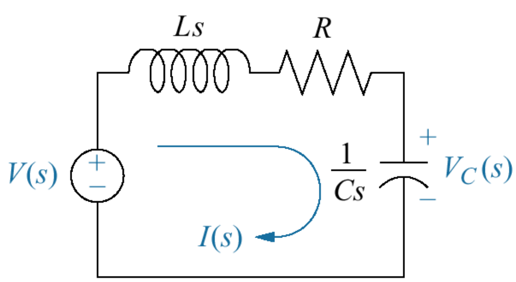
$$LC\frac{d^2v_c(t)}{dt^2} + RC\frac{dv_c(t)}{dt} + v_c(t) = v(t)$$

$$(LCs^2 + RCs + 1)V_c(s) = V(s)$$



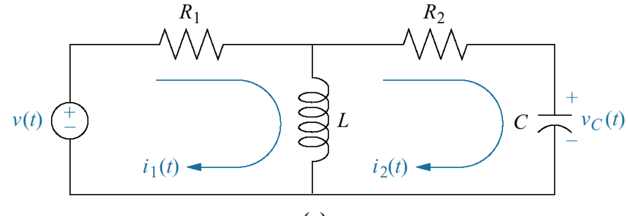
$$\frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

- **The problem can also be simplified by transforming the circuit into the Laplace s-domain.**

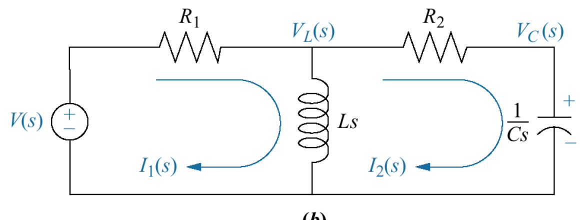


Using the voltage division technique to get the required transfer function.

- **For multiple loop network:**



Transform the circuit into *s*-domain:



Depending on the number of loops in the circuit, obtain the simultaneous equations according to this rule:

[Σ imp Mesh1]I_1(s) - [Σ common imp]I_2(s) = [Σ voltage applied Mesh1]

- [Σ common imp]I_1(s) + [Σ imp Mesh2]I_2(s) = [Σ voltage applied Mesh2]

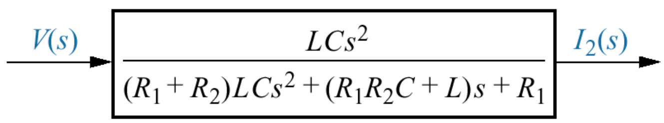
ie,

$$\begin{bmatrix} R_1 + Ls & -Ls \\ -Ls & Ls + R_2 + \dfrac{1}{Cs} \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} V(s) \\ 0 \end{bmatrix}$$

Cramer's rule (or any other method) can be used to solve the simultaneous equations as required:

$$I_2(s) = \frac{\begin{vmatrix} R_1 + Ls & V(s) \\ -Ls & 0 \end{vmatrix}}{\begin{vmatrix} R_1 + Ls & -Ls \\ -Ls & Ls + R_2 + \dfrac{1}{Cs} \end{vmatrix}}$$

to give



- **For three-loop network,**