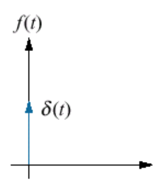
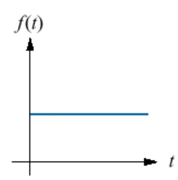
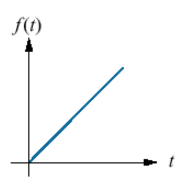
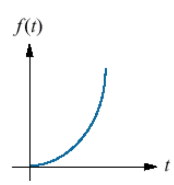
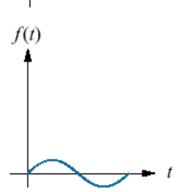
CHAPTER 4 : TIME DOMAIN RESPONSE ANALYSIS

A) Introduction

- **Next important step after a mathematical model of a system is obtained.**

- **To analyze the system's performance.**

- **Normally use the standard input signals to identify the characteristics of system's response (based on situation where the system is being used); standard input signals include –**
  - **Step function**
  - **Ramp function**
  - **Impulse function**
  - **Parabolic function**
  - **Sinusoidal function**

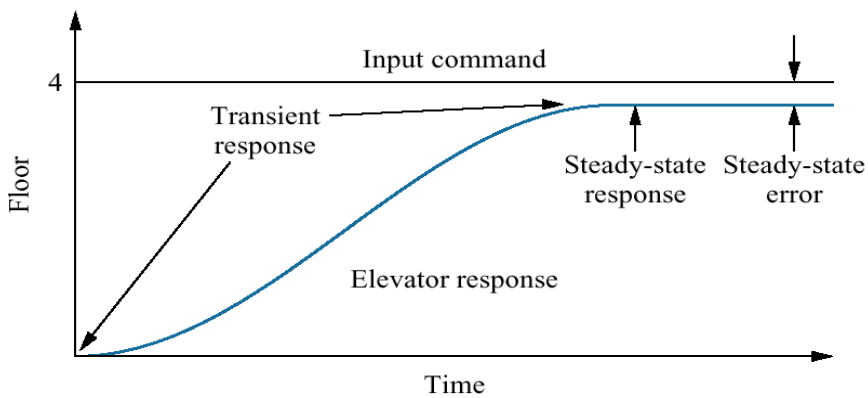
Input	Function	Description	Sketch	Use
Impulse	$\delta(t)$	$\delta(t) = \infty$ for $0- < t < 0+$ $= 0$ elsewhere $\int_{0-}^{0+} \delta(t)\,dt = 1$		Transient response Modeling
Step	$u(t)$	$u(t) = 1$ for $t > 0$ $= 0$ for $t < 0$		Transient response Steady-state error
Ramp	$tu(t)$	$tu(t) = t$ for $t \geq 0$ $= 0$ elsewhere		Steady-state error
Parabola	$\frac{1}{2}t^2u(t)$	$\frac{1}{2}t^2u(t) = \frac{1}{2}t^2$ for $t \geq 0$ $= 0$ elsewhere		Steady-state error
Sinusoid	$\sin \omega t$			Transient response Modeling Steady-state error

B) <u>Transient response and steady state response</u>

- **Output response consists of the sum of *forced response* (form the input) and *natural response* (from the nature of the system).**

- **The natural response determines how good the system is.**

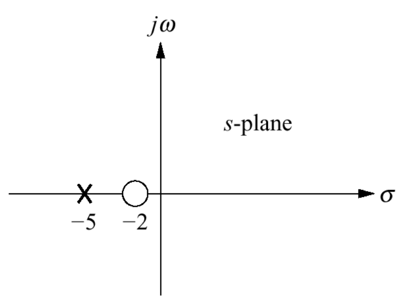
- **As described earlier, the transient response is *the change in output response from the beginning of the response to the final state of the response* and the steady state response is *the output response as time is approaching infinity (or no more changes at the ouput)***



- **Example: For a system with transfer function**

$$\frac{C(s)}{R(s)} = \frac{s+2}{s+5}$$

We can plot the pole and zero of the system as follows:



When a unit step function is applied to the system, we have

$$R(s) = \frac{1}{s}$$

Substitute this input into the transfer function and applying the partial fraction, gives

$$C(s) = \frac{s+2}{s+5}.\frac{1}{s}$$
$$= \frac{2}{5}.\frac{1}{s} + \frac{3}{5}.\frac{1}{s+5}$$

Applying the inverse Laplace transform, gives the output response

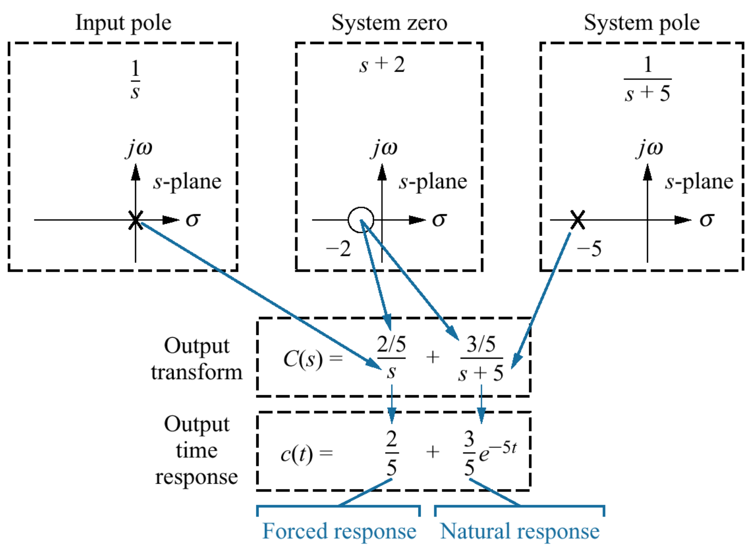
$$c(t) = \frac{2}{5} + \frac{3}{5}e^{-5t}$$

The output response above shows the transient response of the system as the time is varied.

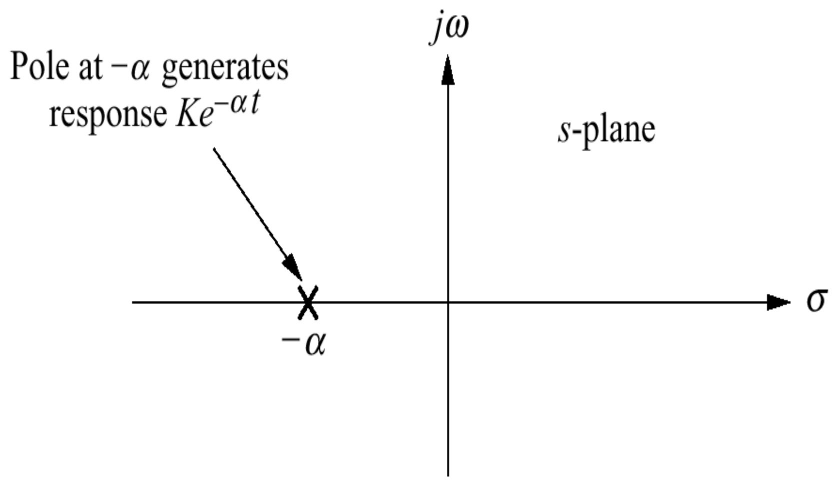
The steady state response can be obtained by putting *t* to infinity, which will give

$$c(\infty) = \frac{2}{5}$$

Notes:



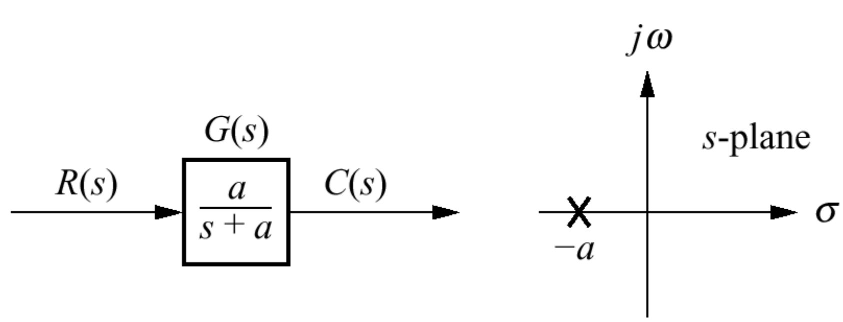
- **Any input to a system will give the forced response at the output**
- **The poles in the transfer function of a system will give the natural response at the output**
- **The poles in the transfer function of a system will give the $e^{-\alpha t}$ term, which is associated with the pole location on *s*-plane. The further away (to the negative side of the real axis) the pole location, the faster the $e^{-\alpha t}$ term goes to 0.**



- **The combination of zeros and poles of a system contributes to the magnitude of the output response.**

C) <u>First order systems</u>

- **A system with only one pole.**



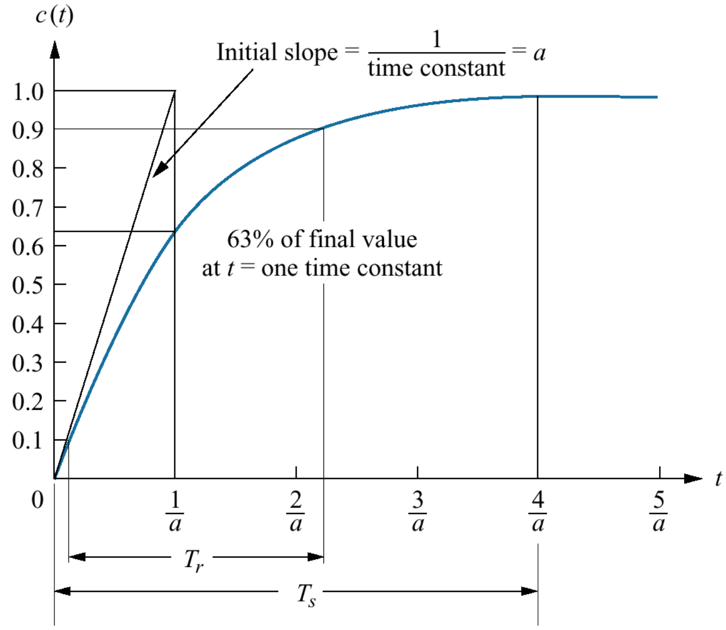
- **Applying unit step function to a first order system above, gives**

$$C(s) = \frac{a}{s+a} R(s)$$
$$= \frac{a}{s+a} \cdot \frac{1}{s}$$
$$= \frac{1}{s} - \frac{1}{s+a}$$

Taking the inverse Laplace transform, yields

$$c(t) = 1 - e^{-at}$$

Where the output response is in the form below



- **Several definitions and specifications can be obtained from the output response of a first order system:**

  - **Time constant, T_c: is the time for the $e^{-\alpha t}$ to decay to 37% of its initial value (or rise to 63% of its final value), or the time when $t = \frac{1}{a}$, ie**

$$T_c = \frac{1}{a}$$

- **Rise time, T_r: the time taken for the output waveform to go from 10% to 90% of its final output value, ie**

$$T_r = \frac{2.2}{a} = 2.2T_c$$

- **Settling time, T_s: the time taken for the output waveform to reach, and stay within 2% of its final output value, ie**

$$T_s = \frac{4}{a} = 4T_c$$

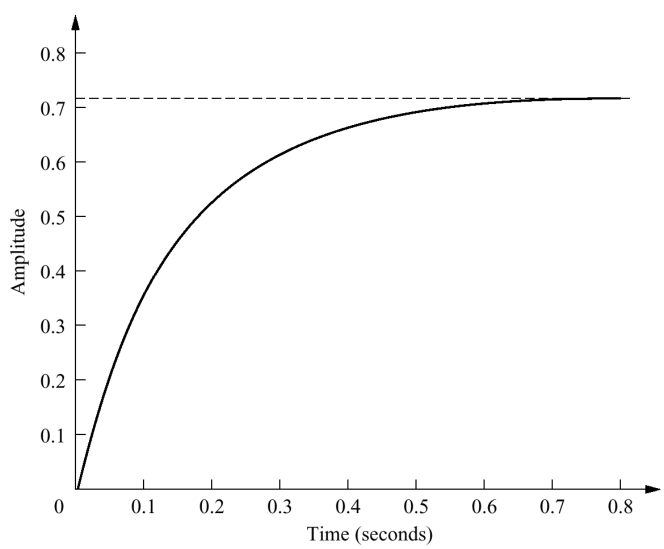
- **In some cases, it is hard to obtain a system's transfer function analytically.**

- **We could obtain the transfer function through experiment or testing.**

- **For example, a simple general first order system would have a transfer function of**

$$\frac{C(s)}{R(s)} = \frac{K}{s+a}$$

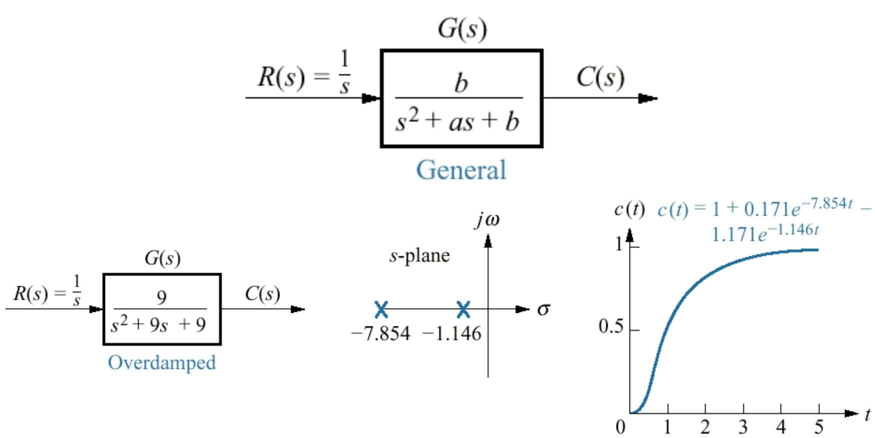
After applying a unit step function, the output response waveform is obtained as shown below,

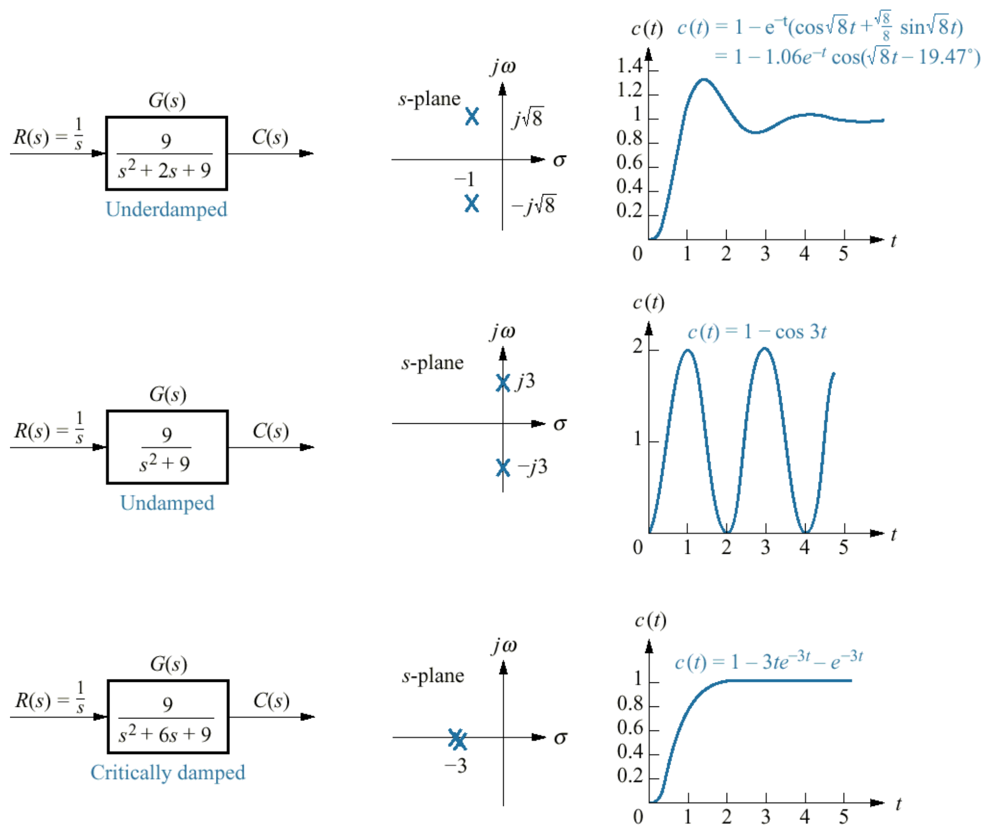


From the output waveform, we could determine the time constant when the output rise to 63% of its final value, which is in this case is $0.63 \times 0.72 = 0.45$. This is about 0.13 second (from the graph). Hence $a = 1/0.13 = 7.7$

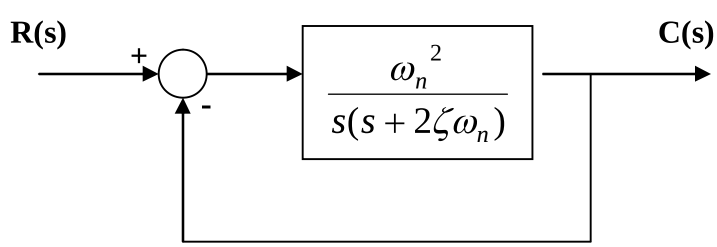
D) <u>Second order systems</u>

- **A system with two poles.**
- **When tested with a unit step input, the second order system will give several type of output response, which we can analyze.**
- **This will depend on the location on the system's poles.**
- **Consider the following example of a second order system:**





- **In general, a second order can be represented by**



The second order response can be obtained from the general closed loop transfer function,

$$\frac{C(s)}{R(s)} = \frac{K\omega_n^{\,2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\,2}}$$

where, ω_n **= undamped natural frequency**

ζ **= damping ratio**

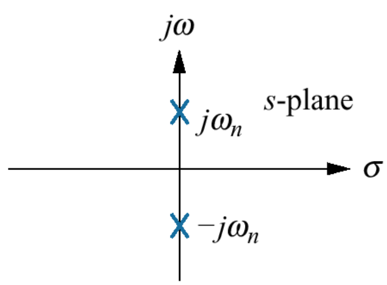
- **The poles for the second order system *C(s)/R(s)* can be found from the denominator as,**

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

- ζ **in the poles equation determines the system output response. If the gain *K* is set to 1, several output response can be obtained.**

- **<u>For ζ = 0</u>, we will have the poles location at**

$$s_{1,2} = \pm j\omega_n$$



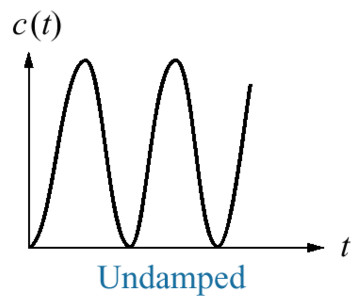
Applying unit step function at the input gives the output as

$$C(s) = \frac{\omega_n^2}{s^2 + \omega_n^2} \cdot \frac{1}{s}$$

Using the partial fraction technique and taking the inverse Laplace transform, gives a general output response for ζ **= 0,**

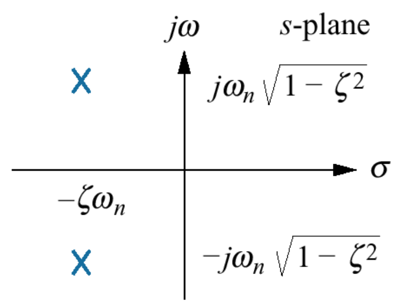
$$c(t) = 1 - A\cos(\omega_n t - \phi)$$

where the output response is oscillating and in an *UNDAMPED* condition (or in a critically stable condition).



- **<u>For 0 < ζ < 1</u>, we will have the poles location at**

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$



Applying unit step function at the input gives the output as

$$C(s) = \frac{\omega_n^{\,2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\,2}}.\frac{1}{s}$$

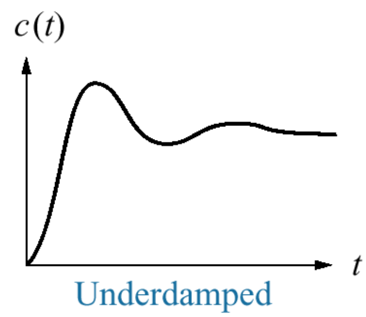
Using the partial fraction technique and taking the inverse Laplace transform, gives a general output response for 0 < ζ < 1,

$$c(t) = 1 - Ae^{-\sigma_d t}\cos(\omega_d t + \theta)$$

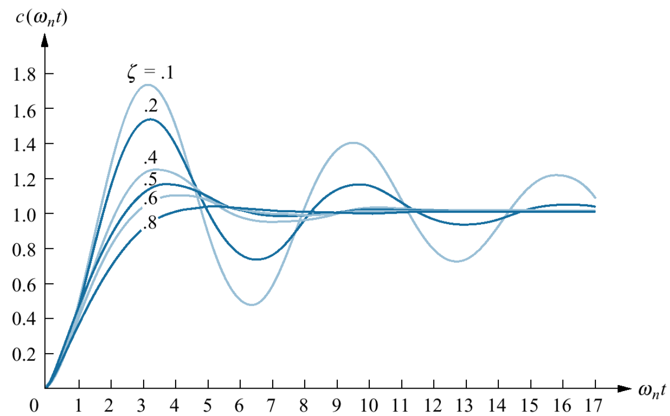
where, $\sigma_d = \zeta\omega_n$ **and** $\omega_d = \omega_n\sqrt{1-\zeta^2}$ **.**

The output response will oscillate (due to the sinusoidal term) and decay out (due to the exponential term).

In this case, the output response is in UNDERDAMPED condition.

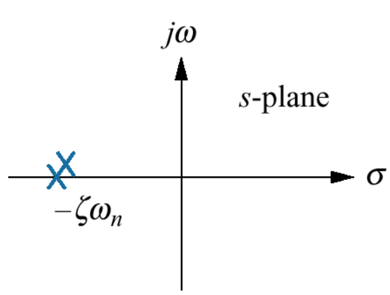


For different value of ζ in this case, a different type of output response will be obtained,



- **<u>For ζ = 1</u>, we will have the poles location at**

$$s_{1,2} = -\omega_n$$



Applying unit step function at the input gives the output as

$$C(s) = \frac{\omega_n^{\ 2}}{s^2 + 2\omega_n s + \omega_n^{\ 2}} . \frac{1}{s}$$

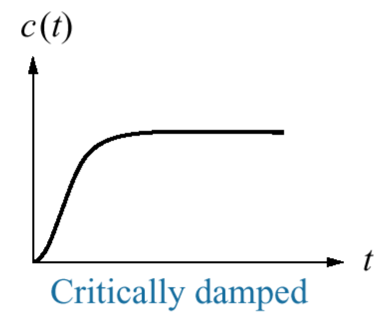
$$= \frac{\omega_n^{\ 2}}{(s + \omega_n)^2} . \frac{1}{s}$$

Using the partial fraction technique and taking the inverse Laplace transform, gives a general output response for ζ = 1,

$$c(t) = 1 - (K_1 e^{-\omega_n t} + K_2 t e^{-\omega_n t})$$

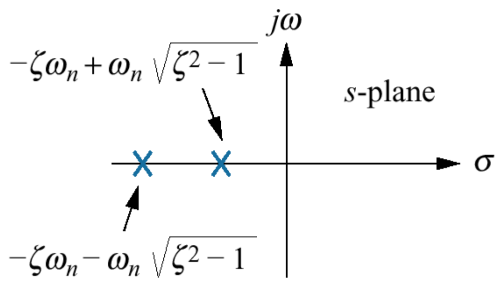
The output response will not oscillate and decay out (quickly without any overshoot or oscillation).

In this case, the output response is in a CRITICALLY DAMPED condition.



- **<u>For $\zeta > 1$</u>, we will have the poles location at**

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$



the poles are negative, real and different to each other

Applying unit step function at the input gives the output as

$$C(s) = \frac{\omega_n^{\ 2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}}.\frac{1}{s}$$

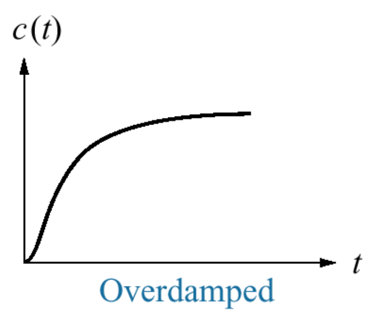
$$= \frac{\omega_n^{\ 2}}{(s + \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1})(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1})}.\frac{1}{s}$$

Using the partial fraction technique and taking the inverse Laplace transform, gives a general output response for ζ > 1,

$$c(t) = 1 - (K_1 e^{-s_1 t} + K_2 e^{-s_2 t})$$

The output response decays out slowly due to the fact that one of the poles is near to the imaginary axis.

In this case, the output response is in an OVERDAMPED condition.



- **The output response of the second order system can be summarized as given below,**